§ 1 The Algebra of Complex Numbers

1.1 Arithmetic Operations

• The field $\mathbb{C}$ of complex numbers is obtained by adjoining the *imaginary unit i* to the field $\mathbb{R}$ of real numbers. The defining property of i is

$$i^2 + 1 = 0$$

A complex number has the form

$$z = a + ib \tag{1}$$

where $(a, b) \in \mathbb{R}^2$, the $+$ symbol represents the process of addition, and ib is a shortened notation for the multiplication of i and b.

a in Eq. (1) is called the *real part* of z, written $a = Re\,(z)$.

b in Eq. (1) is called the *imaginary part* of z, written $b = Im\,(z)$.

If $b = Im\,(z) = 0$, z is said to be *purely real*. If $a = Re\,(z) = 0$, z is said to be *purely imaginary*.

Two complex numbers z_1 and z_2 are equal iff

$$\begin{cases} Re\,(z_1) & = Re\,(z_2) \\ Im\,(z_1) & = Im\,(z_2) \end{cases} \tag{2}$$

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, with $(a_1, a_2, b_1, b_2) \in \mathbb{R}^4$.

• Addition in $\mathbb{C}$ is defined as follows.

$$z_1 + z_2 = (a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2) \tag{3}$$

• Multiplication in $\mathbb{C}$ is defined in the following way.

$$z_1 z_2 = (a_1 + ib_1)\,(a_2 + ib_2) = (a_1 a_2 - b_1 b_2) + i(a_1 b_2 + a_2 b_1) \tag{4}$$

Field structure of $\mathbb{C}$

• It is clear from the above that $\mathbb{C}$ is closed under addition and multiplication

• It is also easy to check that addition and multiplication are associative and commutative, and that multiplication is distributive over addition

• One can easily verify that the identity element for addition is $0 = 0 + i \cdot 0$. The identity element for multiplication is $1 = 1 + i \cdot 0$

• For any complex number $z = a + ib$, $-z = -a + i(-b)$ is its additive inverse. What about the inverse for multiplication? To find out, let us take a small detour and construct the operation of division in $\mathbb{C}$.

Let $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$, with $z_2 \neq 0$. If z_1/z_2 is a complex number $z_3 = a_3 + ib_3$, then

$$a_1 + ib_1 = (a_2 + ib_2)\,(a_3 + ib_3) = (a_2 a_3 - b_2 b_3) + i\,(a_2 b_3 + a_3 b_2)$$

$$\iff \begin{cases} a_1 = a_2 a_3 - b_2 b_3 \\ b_1 = a_2 b_3 + a_3 b_2 \end{cases}$$

This is a linear system of equations, which has a unique solution since $a_2^2 + b_2^2 \neq 0$. The solution is

$$\begin{cases} a_3 = \dfrac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} \\ b_3 = \dfrac{a_2b_1 - a_1b_2}{a_2^2 + b_2^2} \end{cases}$$

Hence

$$\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2} = \frac{a_1a_2 + b_1b_2}{a_2^2 + b_2^2} + i\frac{a_2b_1 - a_1b_2}{a_2^2 + b_2^2} \tag{5}$$

It is now clear that any nonzero complex number $z = a + ib$ has a multiplicative inverse, given by

$$\frac{1}{z} = \frac{a - ib}{a^2 + b^2} \tag{6}$$

We have now completed our verification of the field structure of $\mathbb{C}$.

Important operations

1.4 Conjugation and Modulus

Definition For any complex number $z = a + ib$, the complex number $w = a - ib$, is called *the complex conjugate* of z, written $w = \overline{z}$.

• Conjugation is an *involution:* $\overline{\overline{z}} = \overline{w} = z$.

• The real and imaginary parts of a complex number z can be expressed in terms of z and $\overline{z}$ only:

$$Re\,(z) = \frac{z + \overline{z}}{2} \quad , \quad Im\,(z) = \frac{z - \overline{z}}{2i} \tag{7}$$

• It is easy to verify that if z_1 and z_2 are two complex numbers,

$$\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2 \quad , \quad \overline{z_1z_2} = \overline{z}_1\overline{z}_2 \tag{8}$$

If $z_2 \neq 0$, the latter implies

$$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z}_1}{\overline{z}_2}$$

Application: Consider the polynomial equation

$$a_nz^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0 = 0$$

Let w be a solution to that equation. Then $\bar{w}$ solves

$$\bar{a}_nz^n + \bar{a}_{n-1}z^{n-1} + \cdots + \bar{a}_1z + \bar{a}_0 = 0$$

This proves that the nonreal roots of a polynomial equation with real coefficients occur in complex conjugate pairs.

Definition For any complex number $z = a + ib$, $z\bar{z} = a^2 + b^2 \geq 0$. We define $z\bar{z} = |z|^2$, where $|z| = \sqrt{a^2 + b^2}$ is called the *modulus of z.*

• Note that $|\bar{z}| = |z|$, and the multiplicative inverse of any $z \neq 0$ can thus be written in the concise form

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2} \tag{9}$$

• For all $z \in \mathbb{C}$, $-|z| \leq Re\,(z) \leq |z|$, and $-|z| \leq Im\,(z) \leq |z|$.

• Let $(z_1,\, z_2) \in \mathbb{C}^2$

$$|z_1 z_2|^2 = z_1 z_2 \overline{z_1 z_2} = z_1 \bar{z}_1 z_2 \bar{z}_2 = |z_1|^2 |z_2|^2$$

from which we conclude that

$$|z_1 z_2| = |z_1||z_2| \tag{10}$$

and, if $z_2 \neq 0$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \tag{11}$$

• Let $(z_1,\, z_2) \in \mathbb{C}^2$

$$|z_1 + z_2|^2 = (z_1 + z_2)\left(\overline{z_1 + z_2}\right) = z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2 = |z_1|^2 + |z_2|^2 + 2Re\,(z_1\bar{z}_2) \leq (|z_1| + |z_2|)^2$$

Hence,

$$|z_1 + z_2| \leq |z_1| + |z_2| \tag{12}$$

This is the well-known triangle inequality in the context of complex numbers. We will see its geometric interpretation shortly.

1.2 Square Roots

Definition The *square root* $w = x + iy$ of any complex number $z = a + ib$ must satisfy

$$(x + iy)^2 = z \iff \begin{cases} x^2 - y^2 & = a \\ 2xy & = b \end{cases}$$

Combining the two equations in the system, we find

$$a^2 + b^2 = \left(x^2 - y^2\right)^2 + 4x^2y^2 = \left(x^2 + y^2\right)^2$$

Hence $x^2 + y^2 = \sqrt{a^2 + b^2}$, and from the first equation in the system, we have

$$x^2 = \frac{1}{2}\left(a + \sqrt{a^2 + b^2}\right) \quad , \quad y^2 = \frac{1}{2}\left(-a + \sqrt{a^2 + b^2}\right)$$

and from the second equation $2xy = b$, we must select x and y so that their product xy has the sign of b. This leads to the general solution :

$$w = x + iy = \sqrt{a + ib} = \pm\left(\sqrt{\frac{a + \sqrt{a^2 + b^2}}{2}} + i\,\frac{b}{|b|}\sqrt{\frac{-a + \sqrt{a^2 + b^2}}{2}}\right), \quad b \neq 0 \tag{13}$$

The situation $b = 0$ corresponds to the square roots of real numbers, and the expressions are

$$w = \pm\sqrt{a} \quad \text{if } a \geq 0 \quad , \quad w = \pm i\,\sqrt{-a} \quad \text{if } a < 0, \quad b = 0 \tag{14}$$

The square root of any complex number has two opposite values, which coincide only if $a + ib = 0$.

§ 2 The Geometric Representation of Complex Numbers

2.1 Geometric Addition and Multiplication

• Complex numbers can be represented as vectors in $\mathbb{R}^2$ in the Cartesian way, the x-axis corresponding to the real part of the number, and the y-axis corresponding to the imaginary part. The plane with orthogonal axes given by the real part and the imaginary part is called the *complex plane*.

The addition of two complex numbers can be viewed as vector addition in the complex plane. The inequality $|z_1 + z_2| \leq |z_1| + |z_2|$ has a clear meaning in this context, since $|z_1|$, $|z_2|$ and $|z_1 + z_2|$ are the lengths of all 3 sides of the triangle in Figure 1.

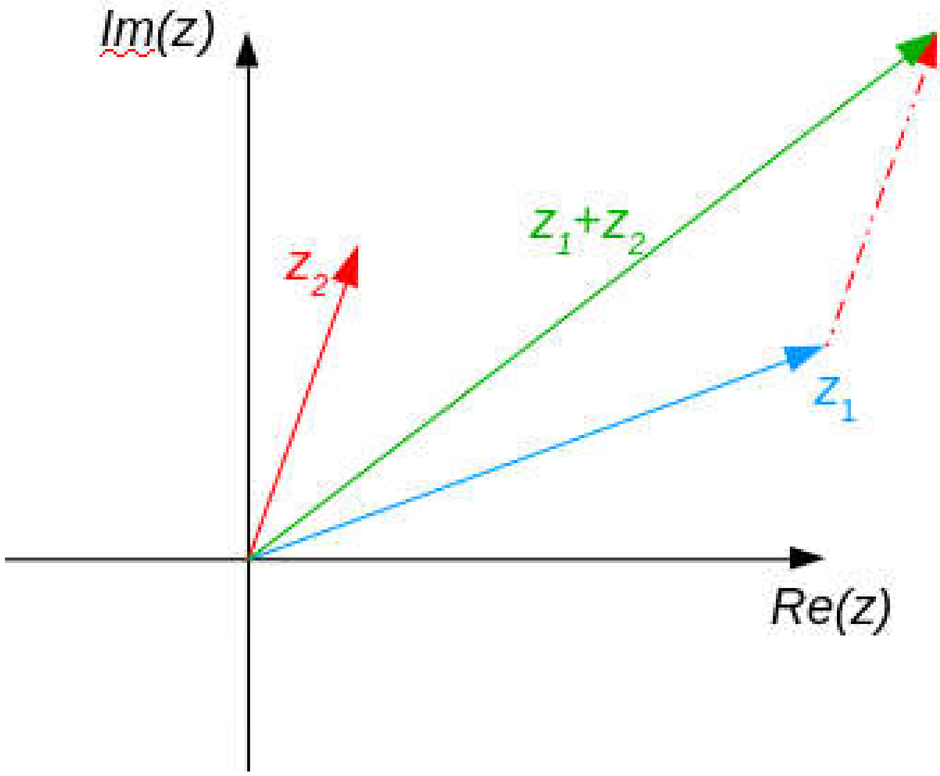

Figure 1: Addition in the complex plane

In the complex plane, z and $\bar{z}$ lie symmetrically with respect to the real axis.

• *Re* (z) and *Im* (z) are the Cartesian coordinates of the representation of z in the complex plane. One may also consider polar coordinates $(r,\, \theta)$ to represent z, i.e. such that

$$z = a + ib = r\cos\theta + ir\sin\theta = r\,(\cos\theta + i\sin\theta)$$

Clearly, $r = |z|$. The polar angle θ is called the *argument* of z.

Now, let $z_1 = r_1\,(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2\,(\cos\theta_2 + i\sin\theta_2)$

$$\begin{aligned} z_1 z_2 &= r_1 r_2\,[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i\ (\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2)] \\ &= r_1 r_2\,[\cos(\theta_1 + \theta_2) + i\ \sin(\theta_1 + \theta_2)] \end{aligned}$$

Complex numbers are multiplied by multiplying their lengths and adding their angles.

2.2 The Binomial Equation

• The polar representation is particularly convenient to evaluate the powers of a complex number z. Indeed, we have just shown that for $z = r(\cos\theta + i\sin\theta)$,

$$z^n = r^n [\cos(n\theta) + i\sin(n\theta)] \tag{15}$$

For $|z| = 1$, we recognize the well-known *de Moivre's formula*:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta) \tag{16}$$

Observe also that from the equality $1/z = \bar{z}/|z|^2$, we have

$$\frac{1}{z} = \frac{1}{r}(\cos\theta - i\sin\theta) = \frac{1}{r}[\cos(-\theta) + i\sin(-\theta)] = r^{-1}[\cos(-\theta) + i\sin(-\theta)]$$

so that $z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$ holds also for negative integers n.

• The n^{th} root w of a complex number $z = r(\cos\theta + i\sin\theta)$ can be readily computed using this representation:

Let $w = \rho(\cos\gamma + i\sin\gamma)$, with $(\rho, \gamma) \in \mathbb{R}^+ \times \mathbb{R}$, such that

$$w^n = z \iff \rho^n[\cos(n\gamma) + i\sin(n\gamma)] = r(\cos\theta + i\sin\theta)$$

We obtain the n different n^{th} roots of z by setting

$$\begin{cases} \rho &= \sqrt[n]{r} \\ \gamma &= \dfrac{\theta}{n} + \dfrac{2\pi k}{n} \end{cases}, \quad k = 0, 1, \ldots, n-1 \tag{17}$$

i.e.

$$w = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right] \quad , \ k = 0, 1, \ldots, n-1$$

In the complex plane, the n w's are the vertices of a regular polygon with n sides.

• The well-known case of the n^{th} roots of unity correspond to $z = 1$. If we set

$$w = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$$

then the n^{th} roots of 1 are $1, w, w^2, \ldots, w^{n-1}$.

2.4 The Spherical Representation

• We will soon see that it is often useful to consider the set $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The symbol ∞ represents infinity, and is connected to finite numbers in the following way:

(1) for all finite z, $z + \infty = \infty + z = \infty$;

(2) for all $z \neq 0$ in $\widehat{\mathbb{C}}$, $z \cdot \infty = \infty \cdot z = \infty$. Furthermore, we adopt the convention that for $z \neq 0$ in $\widehat{\mathbb{C}}$, $z/0 = \infty$, and for $z \in \mathbb{C}$, $z/\infty = 0$.

• ∞ cannot be represented in the standard complex plane. One may however consider an extended complex plane, containing a point at infinity, which is such that every straight line passes through it, but no half plane contains it.

There exists an alternative geometric representation for complex numbers in which every element of $\widehat{\mathbb{C}}$ has a concrete representation, and in which the extended complex plane introduced above makes intuitive sense. This representation is based on the *stereographic projection*, which is constructed as follows.

- Consider the unit sphere S in $\mathbb{R}^3$ given by $x_1^2 + x_2^2 + x_3^2 = 1$
- To every point $Z = (x_1,\, x_2,\, x_3)$ in S except for $(0,\, 0,\, 1)$, assign $w \in \mathbb{C}$ according to the transformation

$$w = \frac{x + iy}{1 - x_3} \tag{18}$$

The transformation is one-to-one, as can be shown by straightforward algebra:

$$|w|^2 = \frac{x^2 + y^2}{(1 - x_3)^2} = \frac{1 + x_3}{1 - x_3} \iff x_3 = \frac{|w|^2 - 1}{|w|^2 + 1}$$

Likewise, simple computations give

$$x = \frac{w + \bar{w}}{1 + |w|^2}, \quad y = -i\, \frac{w - \bar{w}}{1 + |w|^2}$$

- If one assigns ∞ to $(0,\, 0,\, 1)$, we have an *homeomorphism* between S and $\widehat{\mathbb{C}}$.

• Now, if we identify the x-y plane with the complex plane, the transformation defined by Eq. (18) has a simple geometric interpretation. Consider the point N, with coordinates $(0,\, 0,\, 1)$, and any point $Z = (x,\, y,\, z)$ on S. Then w is the point which corresponds to the intersection of the line in $\mathbb{R}^3$ through N and Z with the x-y plane, as shown in Figure 2.

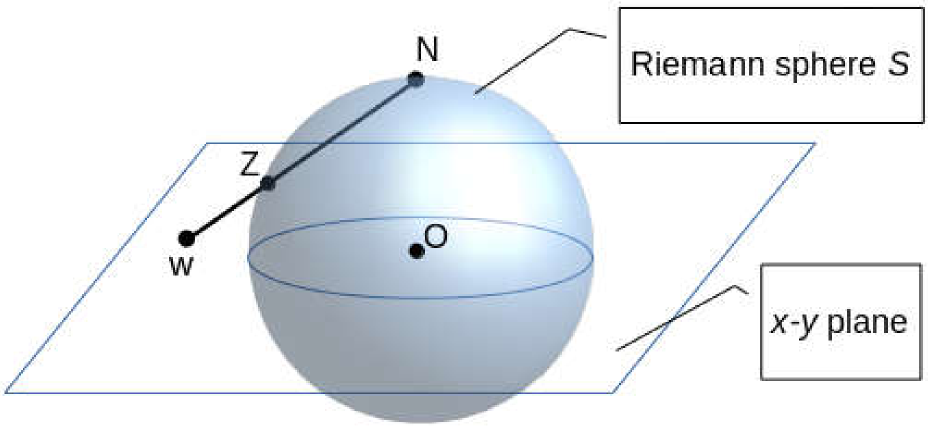

Figure 2: Schematic of the stereographic projection

Let us prove this. A parametric equation for the line through Z and N in $\mathbb{R}^3$ is

$$r(t) = (0,\, 0,\, 1) + t\,(x_1,\, x_2,\, x_3 - 1)$$

The line intersects the x-y plane for t such that

$$1 + t\,(x_3 - 1) = 0 \iff t = \frac{1}{1 - x_3}$$

This corresponds to the point in $\mathbb{R}^3$ with coordinates

$$r\left(\frac{1}{1 - x_3}\right) = \left(\frac{x_1}{1 - x_3},\, \frac{x_2}{1 - x_3},\, 0\right)$$

As we already mentioned, the bijective projection from any point Z on the sphere- except for N- to any point w in the x-y plane (or complex plane) is called a *stereographic projection*.

• You can convince yourself, first geometrically and then algebraically, that every straight line in the x-y plane is transformed into a circle on S which passes through N by the stereographic projection.

More generally, any circle on S is mapped either to a circle or to a line in the x-y plane.